

Integrable open spin chains related to infinite matrix product states

B. Basu-Mallick*

Theory Division, Saha Institute of Nuclear Physics, 1/AF Bidhan Nagar, Kolkata 700 064, India

F. Finkel[†] and A. González-López[‡]

Departamento de Física Teórica II, Universidad Complutense de Madrid, 28040 Madrid, Spain

(Dated: April 8, 2016)

In this paper we study an $\text{su}(m)$ -invariant open version of the Haldane–Shastry spin chain whose ground state can be obtained from the chiral correlator of the $c = m - 1$ free boson boundary conformal field theory. We show that this model is integrable for a suitable choice of the chain sites depending on the roots of the Jacobi polynomial $P_N^{\beta-1, \beta'-1}$, where N is the number of sites and β, β' are two positive parameters. We also compute in closed form the first few nontrivial conserved charges arising from the twisted Yangian invariance of the model. We evaluate the chain's partition function, determine the ground state energy and deduce a complete description of the spectrum in terms of Haldane's motifs and a related classical vertex model. In particular, this description entails that the chain's level density is normally distributed in the thermodynamic limit. We also analyze the spectrum's degeneracy, proving that it is much higher than for a typical Yangian-invariant model.

PACS numbers: 75.10.Pq, 02.30.Ik, 11.25.Hf

I. INTRODUCTION

Recent experiments involving optical lattices of ultra-cold Rydberg atoms and trapped ions, neutral atoms in optical cavities, etc., offer the possibility of realizing various theoretical models of lower-dimensional spin systems with long-range interactions in a remarkably precise way^{1–5}. For example, by using hyperfine ‘clock’ states of trapped $^{171}\text{Yb}^+$ ions it has become possible to realize one-dimensional spin systems with tunable long-range interactions, where the coupling between the i -th and j -th lattice sites falls off approximately algebraically as $J_{ij} \propto 1/|i-j|^\alpha$, with $\alpha \in (0, 3)$. Furthermore, it has been found that, unlike the case of spin chains with nearest or next-to-nearest neighbor interactions, spin chains with long-range interactions often exhibit interesting physical phenomena like realization of quantum spin glasses, quantum crystals and high-speed propagation of correlations exceeding light-cone-like bounds^{2,3,6,7}.

Due to these salient features of strongly correlated lower-dimensional systems, the theoretical investigation of exactly solvable and quantum integrable spin chains with long-range interactions has acquired great impetus. The study of this type of quantum integrable spin systems was pioneered by Haldane and Shastry^{8,9}, who found the exact spectrum of a circular array of equispaced $\text{su}(2)$ spins with two-body interactions inversely proportional to the square of their chord distances. This Haldane–Shastry (HS) spin chain has many remarkable properties: to name only a few, its exact ground state wave function coincides with the $U \rightarrow \infty$ limit of Gutzwiller's variational wave function for the Hubbard model^{10,11}, and its spinon excitations obey a generalized Pauli exclusion principle¹². Furthermore, this chain exhibits Yangian quantum group symmetry, due to which the corresponding spectrum can be expressed in closed

form by using the so called ‘motifs’^{13,14}.

In the past few years, infinite matrix product states (MPS) related to $(1+1)$ -dimensional conformal field theories (CFT) have been used to construct HS-like quantum spin chains with $\text{su}(m)$ spin and periodic boundary conditions^{15–18}. In this approach, finite-dimensional matrices associated with MPS are replaced by chiral vertex operators of a CFT, and the corresponding correlator is interpreted as the ground state wave function of the spin system. Very recently, an inhomogeneous open version of the HS spin chain has been constructed by using infinite MPS from a suitable boundary conformal field theory¹⁹. In the $\text{su}(2)$ case this construction naturally yields a linear system determining the two-point spin correlation functions, which can be solved in closed form for a particular (uniformly distributed) choice of the chain sites. In fact, the previous reference mainly focus on three instances of chains with equispaced sites, for which they discuss the integrability, conjecture a formula for the spectrum from numerical computations and determine the twisted Yangian generators responsible for the huge spectral degeneracy. The purpose of the present paper is twofold. In the first place, we shall show that these three equispaced chains can be embedded into a large class of integrable open spin chains whose lattice sites (no longer equally spaced) depend on two free parameters. We shall then compute the first few nontrivial conserved charges arising from the model's twisted Yangian invariance, and show that they coincide with the twisted Yangian generators of Ref. 19 in the three equispaced cases. Secondly, we shall evaluate in closed form the partition function of these models, providing a rigorous derivation of the formula for the energy spectrum conjectured in Ref. 19 for the equispaced cases. It should be stressed that our results apply to the whole two-parameter family of integrable spin chains mentioned above, and not just to the

three particular instances thereof studied in Ref. 19. In particular, in the $\text{su}(2)$ case the two-point spin correlators of all of these models are determined by the linear system deduced in the latter reference for the three equispaced cases.

The paper is organized as follows. Section II is devoted to recalling the definition of the $\text{su}(m)$ Simons–Altshuler model and its connection with the $c = m - 1$ free boson boundary CFT. In Section III we construct a two-parameter integrable generalization of the latter model, and show that its ground state is still given by a chiral correlator of a boundary CFT with $m - 1$ free bosons. The derivation of the first few nontrivial conserved charges arising from the twisted Yangian symmetry of the model is presented in Section IV. In Section V we show how to compute in closed form the model’s partition function, from which we deduce its equivalence to a one-dimensional classical vertex model with a simple dispersion relation. This result is used in Section VI to provide a description of the spectrum in terms of Haldane’s motifs. In particular, we compute the ground state energy and prove the formula for the energy spectrum proposed in Ref. 19 for the three equispaced cases. In Section VII we present a brief summary of our work and outline several possible future developments thereof. The paper ends with a short technical appendix, in which we prove an identity for the zeros of Jacobi polynomials needed to compute the twisted Yangian conserved charges.

II. THE $\text{su}(m)$ SIMONS–ALTSHULER MODEL

Consider, to begin with, a spin $1/2$ chain whose fixed sites $z_j = e^{2i\xi_j}$ ($\xi_j \in [0, \pi/2]$, $j = 1, \dots, N$) lie on the upper unit circle in the complex plane, and let $|s_j\rangle$ ($s_j = \pm 1$) be the canonical spin basis at the j -th site. We shall denote the mirror image z_j^* of the lattice site z_j by \bar{z}_j , and shall also set

$$u_j = \frac{1}{2}(z_j + \bar{z}_j) = \cos(2\xi_j).$$

Following Ref. 19, we shall take as ground state of the model under consideration the chiral correlator

$$\psi(s_1, \dots, s_N) = \langle A^{s_1}(u_1) A^{s_2}(u_2) \cdots A^{s_N}(u_N) \rangle, \quad (1)$$

where $A^{s_j}(u_j) = \chi_j : e^{is_j\phi(u_j)/\sqrt{2}} :$ ($: \dots :$ denoting, as usual, the normal ordering), $\phi(u)$ is a chiral bosonic field from the $c = 1$ free boson CFT, and $\chi_j = s_j$ for even j and 1 otherwise. As shown in Ref. 19, ψ is annihilated by the $3N$ operators²⁰

$$\Lambda_k^a = \frac{1}{3\sqrt{2}} \sum_{j(\neq k)} (w_{kj} + w_{k\bar{j}}) \left(s_j^a + i \sum_{b,c=1}^3 \varepsilon_{abc} s_k^b s_j^c \right),$$

with $1 \leq k \leq N$ and $a = 1, 2, 3$, where $\mathbf{s}_i = (s_i^1, s_i^2, s_i^3)$ is the spin operator of the i -th particle, ε_{abc} is the Levi-

Civita symbol, and we have set

$$w_{kj} = \frac{z_k + z_j}{z_k - z_j}.$$

Thus the state $\psi(s_1, \dots, s_N)$ defined above is by construction the ground state of the positive-definite Hamiltonian $\sum_{a,i} (\Lambda_i^a)^\dagger \Lambda_i^a$. Calling

$$h_{ij} = w_{ij}(c_i - c_j) + w_{i\bar{j}}(c_i + c_j), \quad c_j = w_{\bar{j}j} + \sum_{l(\neq j)} (w_{lj} + w_{\bar{l}j}),$$

it can be proved¹⁹ that the latter Hamiltonian can be written as

$$\mathcal{H}_{\text{MPS}} = \sum_{i \neq j} \left(\frac{1}{|z_i - z_j|^2} + \frac{1}{|z_i - \bar{z}_j|^2} - \frac{h_{ij}}{12} \right) \mathbf{s}_i \cdot \mathbf{s}_j \quad (2)$$

up to an additive constant and an unimportant term proportional to the square of the total spin operator $\mathbf{S} \equiv \sum_i \mathbf{s}_i$. For the three particular cases in which the angles of the chain sites are given by

$$\xi_j = \frac{\pi(j - \frac{1}{2})}{2N}, \quad \frac{\pi j}{2N+2}, \quad \frac{\pi j}{2N+1}, \quad (3)$$

it is shown in Ref. 19 that the term h_{ij} in Eq. (2) is a constant (independent of i and j) respectively equal to 0, 4, 2. In these so called *uniform* cases, the model (2) essentially coincides with the integrable open chain of Haldane–Shastry type introduced by Simons and Altshuler²¹, whose Hamiltonian is usually written as²²

$$\begin{aligned} \mathcal{H} &= \sum_{i \neq j} \left(\frac{1}{|z_i - z_j|^2} + \frac{1}{|z_i - \bar{z}_j|^2} \right) (P_{ij} - 1) \\ &= \frac{1}{4} \sum_{i \neq j} \left(\sin^{-2} \xi_{ij}^- + \sin^{-2} \xi_{ij}^+ \right) (P_{ij} - 1). \end{aligned} \quad (4)$$

In the previous formula $\xi_{ij}^\pm = \xi_i \pm \xi_j$, with ξ_j given by Eq. (3), and P_{ij} is the spin exchange operator, defined by

$$P_{ij} |\dots, s_i, \dots, s_j, \dots\rangle = |\dots, s_j, \dots, s_i, \dots\rangle.$$

Indeed, since $P_{ij} = 2 \mathbf{s}_i \cdot \mathbf{s}_j + \frac{1}{2}$ and h_{ij} does not depend on i and j in the three uniform cases, the Hamiltonian (2) differs from $\mathcal{H}/2$ by a constant and an irrelevant term proportional to \mathbf{S}^2 .

In fact, P_{ij} is still defined by the previous formula when the internal spin space is m -dimensional (where now $s_i = -(m-1)/2, -(m-1)/2 + 1, \dots, (m-1)/2$). In this case the exchange operators P_{ij} are related to the $\text{su}(m)$ generators t_k^a in the fundamental representation (where k is the site index, $a = 1, \dots, m^2 - 1$ and $\text{tr}(t_k^a t_k^b) = \frac{1}{2} \delta_{ab}$) by

$$P_{ij} = 2 \sum_{a=1}^{m^2-1} t_i^a t_j^a + \frac{1}{m} \equiv 2 \mathbf{t}_i \cdot \mathbf{t}_j + \frac{1}{m}. \quad (5)$$

With this general definition, the Hamiltonian (4) can be regarded as an $\mathfrak{su}(m)$ spin model. It was shown in Ref. 19 that in this case $\mathcal{H}/2$ differs from the $\mathfrak{su}(m)$ generalization of (2), namely

$$H_{\text{MPS}} = \sum_{i \neq j} \left(\frac{1}{|z_i - z_j|^2} + \frac{1}{|z_i - z_j|^2} - \frac{h_{ij}}{4(m+1)} \right) \mathbf{t}_i \cdot \mathbf{t}_j, \quad (6)$$

by an additive constant and a trivial term proportional to \mathbf{T}^2 , where $\mathbf{T} \equiv \sum_i \mathbf{t}_i$ is the total $\mathfrak{su}(m)$ spin operator. Moreover, as explained in Ref. 19, the ground state of the Hamiltonian H_{MPS} is still given by the chiral correlator (1), where now

$$A^\alpha(u) = \kappa_\alpha : e^{i\mu_\alpha \cdot \phi(u)/\sqrt{2}} :, \quad 1 \leq \alpha \leq m,$$

$\phi \equiv (\phi_1, \dots, \phi_{m-1})$ is a vector of chiral bosonic fields in the $c = m - 1$ free boson boundary CFT. In the latter equation κ_α denotes a Klein factor (commuting with vertex operators and satisfying $\{\kappa_\alpha, \kappa_\beta\} = 2\delta_{\alpha\beta}$) and $\mu_\alpha \in \mathbb{C}^{m-1}$ is the weight vector of the α -th internal state in the fundamental representation of $\mathfrak{su}(m)$.

III. INTEGRABLE GENERALIZATION

We shall show in this section that the $\mathfrak{su}(m)$ model (4) can be greatly generalized (without losing its integrability) by choosing the chain sites so that u_1, \dots, u_N are the N roots of the Jacobi polynomial $P_N^{\beta-1, \beta'-1}$, where β, β' are two positive parameters. In fact, the sites in the uniform cases (3) are obtained when (β, β') respectively take the values $(1/2, 1/2)$, $(3/2, 3/2)$ and $(3/2, 1/2)$.

In order to understand how this integrable generalization comes about, consider to begin with the Hamiltonian of the open Sutherland spin *dynamical* model, given by^{23,24}

$$H_{\text{BC}} = -\Delta + a \sum_{i \neq j} [\sin^{-2} x_{ij}^- (a - P_{ij}) + \sin^{-2} x_{ij}^+ (a - \tilde{P}_{ij})] + \sum_i [b(b - P_i) \sin^{-2} x_i + b'(b' - P_i) \cos^{-2} x_i], \quad (7)$$

where $\Delta = -\sum_i \partial_{x_i}^2$, $x_{ij}^\pm = x_i \pm x_j$, $a > 0$, $b = \beta a$, $b' = \beta' a$, $\tilde{P}_{ij} = P_i P_j P_{ij}$ and P_i is the spin reversal operator, defined by

$$P_i |\dots, s_i, \dots\rangle = |\dots, -s_i, \dots\rangle.$$

We shall consider instead a novel variant of this model, obtained by replacing the spin reversal operators P_i by the identity, which amounts to taking a different representation of the Weyl group of BC_N type. In this way we obtain the Hamiltonian

$$H = -\Delta + a \sum_{i \neq j} (\sin^{-2} x_{ij}^- + \sin^{-2} x_{ij}^+) (a - P_{ij}) + \sum_i [b(b - 1) \sin^{-2} x_i + b'(b' - 1) \cos^{-2} x_i], \quad (8)$$

to which one can associate the auxiliary *scalar* operator

$$H' = -\Delta + a \sum_{i \neq j} [\sin^{-2} x_{ij}^- (a - K_{ij}) + \sin^{-2} x_{ij}^+ (a - \tilde{K}_{ij})] + \sum_i [b \sin^{-2} x_i (b - K_i) + b' \cos^{-2} x_i (b' - K_i)].$$

In the latter equation the operators K_{ij} and K_i act on a scalar function as

$$K_{ij} f(\dots, x_i, \dots, x_j, \dots) = f(\dots, x_j, \dots, x_i, \dots), \\ K_i f(\dots, x_i, \dots) = f(\dots, -x_i, \dots),$$

and $\tilde{K}_{ij} = K_{ij} K_i K_j$. It was shown in Ref. 24 that H' commutes with the family of (commuting) BC_N -type dynamical Dunkl operators $J_k = i \partial_{x_k} + 2ad_k$ ($k = 1, \dots, N$), where

$$d_k = \frac{1}{2} \sum_{l \neq k} [(1 - i \cot x_{kl}^-) K_{kl} + (1 - i \cot x_{kl}^+) \tilde{K}_{kl}] - \sum_{l < k} K_{kl} + \frac{1}{2} [\beta(1 - i \cot x_k) + \beta'(1 + i \tan x_k)] K_k. \quad (9)$$

Equating to zero the coefficient of a^2 in the commutator of H' with J_k we easily arrive at the relation

$$[h'(\mathbf{x}), d_k] = \frac{i}{8} \frac{\partial U}{\partial x_k},$$

where $\mathbf{x} = (x_1, \dots, x_N)$ and

$$U(\mathbf{x}) = \sum_{i \neq j} (\sin^{-2} x_{ij}^- + \sin^{-2} x_{ij}^+) + \sum_i (\beta^2 \sin^{-2} x_i + \beta'^2 \cos^{-2} x_i), \\ h'(\mathbf{x}) = \frac{1}{4} \sum_{i \neq j} [\sin^{-2} x_{ij}^- (1 - K_{ij}) + \sin^{-2} x_{ij}^+ (1 - \tilde{K}_{ij})] + \frac{1}{4} \sum_i (\beta \sin^{-2} x_i + \beta' \cos^{-2} x_i) (1 - K_i).$$

It should be noted that the spin chain Hamiltonian \mathcal{H} in Eq. (4) coincides with the operator $-h(\xi)$, where $h(\mathbf{x})$ is obtained from $h'(\mathbf{x})$ by the formal replacements $(K_{ij}, K_i) \mapsto (P_{ij}, 1)$. Following the approach of Refs. 22 and 24, the integrability condition for the Hamiltonian \mathcal{H} (with chain sites not yet determined) is the vanishing of the commutator $[h'(\mathbf{x}), d_k]$ on the chain sites ξ . Thus a Hamiltonian of the form (4) is integrable provided that its lattice sites satisfy the system of equations

$$\frac{\partial U}{\partial x_k}(\xi) = 0, \quad k = 1, \dots, N.$$

As is shown in Ref. 25, when the parameters β, β' are both positive the latter system has essentially a unique solution determined by the conditions

$$P_N^{\beta-1, \beta'-1}(\cos 2\xi_j) = 0, \quad j = 1, \dots, N. \quad (10)$$

This establishes the integrability of the spin chain (4) with sites satisfying Eq. (10) for arbitrary (positive) values of β, β' . In fact, it should be noted that the parameters (β', β) and (β, β') give rise to the *same* Hamiltonian (4). Indeed, from the equality $P^{\beta, \beta'}(u) = P^{\beta', \beta}(-u)$ it follows that the angles ξ'_j determined by the parameters (β', β) are related to the ξ_j 's in Eq. (10) by $2\xi'_j = \pi - 2\xi_{N-j+1}$, and (4) is clearly invariant under the transformation $\xi_j \mapsto \xi'_j$. Thus we could restrict ourselves, without loss of generality, to the case $\beta \geq \beta'$. It can also be shown that the only models of the form (4) with uniformly spaced sites are precisely the three uniform cases (3).

We have remarked in Section II that the $\text{su}(m)$ Hamiltonian (4) essentially reduces to (6) for the three uniform cases $(\beta, \beta') = (1/2, 1/2), (3/2, 3/2), (3/2, 1/2)$. It is natural to enquire whether this also holds for arbitrary positive values of β and β' . To answer this question, we note that substituting $u_j = \cos(2\xi_j)$ ($j = 1, \dots, N$) in the system satisfied by the zeros of the Jacobi polynomial $P_N^{\beta-1, \beta'-1}(u)$ listed in Eq. (5.2a) of Ref. 26 one obtains the equations

$$\sum_{k(\neq j)} (\cot \xi_{jk}^- + \cot \xi_{jk}^+) = (\beta' - \beta) \cot \xi_j - 2\beta' \cot(2\xi_j), \quad (11)$$

$j = 1, \dots, N$. The relations $i w_{jk} = \cot \xi_{jk}^-$, $i w_{j\bar{k}} = \cot \xi_{jk}^+$ and Eqs. (11) then yield

$$i c_j = (2\beta' - 1) \cot(2\xi_j) + (\beta - \beta') \cot \xi_j,$$

from which one readily obtains

$$h_{ij} \equiv (c_i - c_j) w_{ij} + (c_i + c_j) w_{i\bar{j}} = 2(\beta + \beta' - 1), \quad (12)$$

in agreement with the result for the special values of β and β' mentioned above. Since the right-hand side of (12) is independent of i and j , from Eq. (5) it again follows that the general $\text{su}(m)$ model (4) with sites satisfying (10) with *arbitrary* (positive) β and β' is equivalent to the Hamiltonian (6) (i.e., $\mathcal{H} = 2\mathcal{H}_{\text{MPS}} + A\mathbf{T}^2 + B$ for suitable constants A, B). In particular, this shows that the ground state of the general model (4) coincides with a correlator of the $c = m - 1$ free boson boundary CFT, thus generalizing the results in Ref. 19 for the three uniform cases. For instance, in the $\text{su}(2)$ case this observation implies that the linear system for the two-point correlation functions of the model (2) deduced in Ref. 19 also holds for the more general integrable model (4) with sites determined by the conditions (10).

We shall end this section with a brief discussion of the distribution of the sites of the general model (4) with sites (10). To begin with, it is shown in Ref. 27 that when $N \rightarrow \infty$ the density of the zeros u_j of the Jacobi polynomial $P_N^{\beta-1, \beta'-1}$ in the interval $[-1, 1]$ approaches the continuous distribution

$$\rho(u) = \frac{1}{\pi\sqrt{1-u^2}}.$$

This result easily implies that the chain sites $z_j = e^{2i\xi_j}$ become uniformly distributed on the upper unit circle as N tends to infinity. In fact, according to a classical result of Szegő²⁸, when $N \gg 1$ the angles $2\xi_j$ satisfy

$$2\xi_j = \frac{j\pi}{N} + O(N^{-1}),$$

where $O(N^{-1})$ is uniformly bounded in $j \leq N$. (Note that this property clearly holds in the uniform cases listed in Eq. (3).) In particular, when $1/2 \leq \beta, \beta' \leq 3/2$ we have the more precise estimates²⁸

$$\frac{(2j-1)\pi}{2N+1} \leq 2\xi_j \leq \frac{2j\pi}{2N+1},$$

with equality if and only if $(\beta, \beta') = (1/2, 3/2)$ (in the first inequality) or $(3/2, 1/2)$ (in the second one). In fact, both of these special cases give rise to the same Hamiltonian (4) —the third uniform model in Eq. (3)—, on account of the remark following Eq. (10).

IV. TWISTED YANGIAN SYMMETRY

It was shown in Ref. 22 that the spin 1/2 model (4) with (β, β') in the three uniform cases mentioned above possesses a monodromy matrix $T(u)$ which satisfies the reflection equation²⁹. More precisely, the monodromy matrix is given by²²

$$T(u) = \left(1 + \frac{\beta + \beta'}{2u}\right) \tilde{T}(u),$$

where

$$\tilde{T}(u) = \pi \left[\prod_{i=1}^N \left(1 + \frac{P_{0i}}{u - d_i}\right) \prod_{i=N}^1 \left(1 + \frac{P_{0i}}{u + d_i}\right) \right]$$

and d_i is defined in Eq. (9). In the latter formula the index 0 labels an auxiliary m -dimensional internal space, and the projection operator π is defined by $\pi(x_j) = \xi_j$ and

$$\pi(K_{i_1 j_1} \cdots K_{i_r j_r} K_{l_1} \cdots K_{l_s}) = P_{i_r j_r} \cdots P_{i_1 j_1}.$$

This monodromy matrix is actually valid for $\text{su}(m)$ spin¹⁹ and arbitrary (positive) values of the parameters (β, β') , since it only depends on the expression of the Hamiltonian in terms of permutation operators P_{ij} and the integrability conditions (10). Thus the general $\text{su}(m)$ model (4), with sites satisfying Eq. (10), possesses twisted Yangian symmetry. Since, by construction, $[\mathcal{H}, T(u)] = 0$ for all u , the coefficients of $1/u$ in the Laurent expansion of $T(u)$ form a family of conserved charges for the Hamiltonian \mathcal{H} of the open Haldane–Shastry chain. In fact, since the term $1 + (\beta + \beta')/2u$ is a scalar, the conserved charges can be more directly

obtained by expanding the remaining term in $T(u)$. In other words, if

$$\tilde{T}(u) = 1 + \sum_{n=1}^{\infty} \frac{J_n}{u^n}$$

then $[\mathcal{H}, J_n] = 0$ for all $n \geq 1$. To begin with, from the expansions

$$\frac{1}{u \mp d_i} = \frac{1}{u} \pm \frac{d_i}{u^2} + \frac{d_i^2}{u^3} + O(u^{-4})$$

and the identity

$$P_{0i} = \frac{1}{m} + 2 \sum_a t_0^a t_i^a \quad (13)$$

we easily obtain³⁰

$$J_1 = 2 \sum_i P_{0i} = \frac{2N}{m} + 4 \sum_a t_0^a \sum_i t_i^a.$$

Thus the first-order conserved charges are the total $\mathfrak{su}(m)$ generators T^a , $1 \leq a \leq m^2 - 1$. Similarly, expanding $\tilde{T}(u)$ to second order we obtain

$$\begin{aligned} J_2 &= \left(\sum_{i < j} + \sum_{i > j} + \sum_{i,j} \right) P_{0i} P_{0j} = N + 2 \sum_{i \neq j} P_{0i} P_{0j} \\ &= N + 2 \sum_{i \neq j} P_{ij} P_{0i}, \end{aligned}$$

which yields the second-order conserved charges

$$J_2^0 = \sum_{i \neq j} P_{ij}, \quad J_2^a = \sum_{i \neq j} P_{ij} t_i^a.$$

Both of these charges, however, are trivial, in the sense that they are polynomial functions of the global $\mathfrak{su}(m)$ generators T^a . Indeed,

$$\begin{aligned} J_2^0 &= \frac{N}{m} (N - 1) + 2 \sum_{i \neq j} \sum_a t_i^a t_j^a \\ &= \frac{N}{m} (N - 1) + 2 \sum_a (T^a)^2 - 2 \sum_i \sum_a (t_i^a)^2 \\ &= \frac{N}{m} (N - 1) + 2 \sum_a (T^a)^2 - \frac{N}{m} (m^2 - 1) \\ &= 2 \sum_a (T^a)^2 + \frac{N}{m} (N - m^2). \end{aligned}$$

The conserved charges J_2^a can be simplified using the identity (5) and the relations

$$t^a t^b = \frac{1}{2m} \delta_{ab} + \frac{1}{2} \sum_c (d_{abc} + i f_{abc}) t^c \quad (14)$$

satisfied by the $\mathfrak{su}(m)$ generators in the fundamental representation. Here d_{abc} and f_{abc} are respectively totally symmetric and totally antisymmetric in a, b, c , with

$$\sum_b d_{abb} = 0, \quad \sum_{b,c} d_{abc} d_{bcr} = \frac{m^2 - 4}{m} \delta_{ar} \quad (15)$$

(see, e.g., Ref. 31). After a straightforward calculation one obtains

$$J_2^a = \left(\frac{2N}{m} - \frac{m}{2} \right) T^a + \sum_{b,c} d_{abc} T^b T^c.$$

Similarly, expanding $\tilde{T}(u)$ to third order in $1/u$ and simplifying slightly the result we arrive at the expression

$$\begin{aligned} \frac{J_3}{2} &= \sum_i P_{0i} \pi(d_i^2) + \sum_{i < j} (P_{0i} P_{0j} - P_{0j} P_{0i}) \pi(d_i) \\ &\quad + \frac{1}{2} \left(\sum_{i < j < k} + \sum_{i < j, k} \right) (P_{0i} P_{0j} P_{0k} + P_{0k} P_{0j} P_{0i}) \\ &= \sum_i P_{0i} \pi(d_i^2) + \sum_{i < j} (P_{0j} - P_{0i}) P_{ij} \pi(d_i) \\ &\quad + \left(\sum_{i < j < k} + \sum_{k < i < j} \right) (P_{0k} P_{ik} P_{jk} + P_{0i} P_{ik} P_{ij}) \\ &\quad + \sum_{i < j} (P_{0i} + P_{ij}). \end{aligned}$$

Using again Eq. (13) to identify the coefficients of $t_0^0 \equiv \mathbb{I}$ and t_0^a in the previous formula, and dropping the trivially conserved term $\sum_{i < j} P_{ij}$, we obtain the third-order conserved charges

$$\begin{aligned} J_3^0 &= \sum_i \pi(d_i^2) + \left(\sum_{i < j < k} + \sum_{k < i < j} \right) P_{ik} (P_{jk} + P_{ij}), \\ J_3^a &= \sum_i t_i^a \pi(d_i^2) + \sum_{i < j} (t_j^a - t_i^a) P_{ij} \pi(d_i) + \sum_i (N - i) t_i^a \\ &\quad + \left(\sum_{i < j < k} + \sum_{k < i < j} \right) (t_k^a P_{ik} P_{jk} + t_i^a P_{ik} P_{ij}). \end{aligned}$$

The first of these charges is trivial. Indeed, note first of all that from the coefficient of a^2 in the identity $H' = \sum_i J_i^2$, where $J_k = i\partial_{x_k} + 2ad_k$ (cf. Ref. 24), it follows that $\sum_i \pi(d_i^2) = U(\mathfrak{g})/4$ is a constant. Secondly, from the permutation group identities $P_{ik} P_{jk} = P_{jk} P_{ij} = P_{ij} P_{ik}$ (valid when i, j, k are all distinct) we easily obtain

$$\begin{aligned} &\left(\sum_{i < j < k} + \sum_{k < i < j} \right) P_{ik} (P_{jk} + P_{ij}) \\ &= \frac{1}{3} \sum'_{i,j,k} P_{ik} (P_{jk} + P_{ij}) = \frac{2}{3} \sum'_{i,j,k} P_{ik} P_{jk}, \end{aligned}$$

where the last sum can be shown to be trivially conserved by repeated application of Eqs. (14)-(15). As to

the remaining third-order charges J_3^a , a long but straightforward calculation yields

$$-4J_3^a = \sum_{i \neq j} (w_{ij} - w_{i\bar{j}})^2 t_i^a + \sum_i [(\beta - \beta') w_{i0} + 2\beta' w_{i\bar{i}}]^2 t_i^a - \sum'_{i,j,k} (w_{ij} - w_{i\bar{j}})(w_{jk} + w_{j\bar{k}}) t_i^a P_{ik} P_{ij}.$$

The first two terms in the latter expression can be simplified with the help of Eq. (A.1) and the identity $2w_{i0}w_{i\bar{i}} = 1 + w_{i0}^2$. Dividing the resulting expression for $-4J_3^a$ by the nonzero coefficient $8\beta'(\beta' + 1)/3$ and dropping a trivially conserved term proportional to T^a we finally obtain the equivalent non-trivial conserved charges

$$Q^a = \sum_i (w_{i\bar{i}}^2 + \gamma_1 w_{i0}^2) t_i^a - \gamma_2 \sum'_{i,j,k} (w_{ij} - w_{i\bar{j}})(w_{jk} + w_{j\bar{k}}) t_i^a P_{ik} P_{ij},$$

where the coefficients $\gamma_{1,2}$ are given by

$$\gamma_1 = \frac{(\beta - \beta')(1 + \beta + \beta')}{4\beta'(\beta' + 1)}, \quad \gamma_2 = \frac{3}{8\beta'(\beta' + 1)}.$$

In particular, for the values of (β, β') corresponding to the three uniform cases the previous expression is in agreement³² with Eq. (8) in Ref. 19.

V. PARTITION FUNCTION

We shall next evaluate in closed form the partition function of the $\text{su}(m)$ spin chain (4). The key idea in this respect is to exploit the connection between the latter model and the $\text{su}(m)$ spin Sutherland model (8) by means of the so-called freezing trick^{33–35}. More precisely, when $a \rightarrow \infty$ the Hamiltonians \mathcal{H} and H are related by $H \simeq H_{\text{sc}} - 4a\mathcal{H}$, where H_{sc} is the Hamiltonian of the scalar Sutherland model obtained from H by replacing P_{ij} by 1. From the latter relation it follows that the partition functions Z , Z_{sc} and \mathcal{Z} of the Hamiltonians H , H_{sc} , \mathcal{H} , respectively, are related by³⁵

$$\mathcal{Z}(T) = \lim_{a \rightarrow \infty} \frac{Z(-4aT)}{Z_{\text{sc}}(-4aT)}. \quad (16)$$

Thus the partition function \mathcal{Z} can be evaluated from the spectra of the Hamiltonians H and H_{sc} , which in turn can be derived from that of the auxiliary operator H' following the approach of Ref. 24.

The spectrum of H' can be computed by noting that it acts triangularly on the (non-orthonormal) basis

$$\phi_{\mathbf{n}}(\mathbf{x}) = \phi(\mathbf{x}) e^{2i\mathbf{n} \cdot \mathbf{x}} \quad \mathbf{n} = (n_1, \dots, n_N) \in \mathbb{Z}^N, \quad (17)$$

where $\phi(\mathbf{x}) = \prod_{i < j} |\sin x_{ij}^- \sin x_{ij}^+|^a \cdot \prod_i |\sin x_i|^b |\cos x_i|^{b'}$. More precisely, we introduce a partial ordering \prec in the

basis (17) as follows. Given a multiindex $\mathbf{n} \in \mathbb{Z}^N$, we define the nonnegative and nonincreasing multiindex $[\mathbf{n}]$ by $[\mathbf{n}] = (|n_{i_1}|, \dots, |n_{i_N}|)$, where $|n_{i_1}| \geq \dots \geq |n_{i_N}|$. If $\mathbf{n}, \mathbf{n}' \in [\mathbb{Z}^N]$ are two such multiindices, we shall say that $\mathbf{n} \prec \mathbf{n}'$ if $n_1 - n'_1 = \dots = n_{i-1} - n'_{i-1} = 0$ and $n_i < n'_i$. For arbitrary $\mathbf{n}, \mathbf{n}' \in \mathbb{Z}^N$, we shall say that $\mathbf{n} \prec \mathbf{n}'$ or $\phi_{\mathbf{n}} \prec \phi_{\mathbf{n}'}$ provided that $[\mathbf{n}] \prec [\mathbf{n}']$. With the help of this partial ordering, it can be shown that

$$H' \phi_{\mathbf{n}} = E_{\mathbf{n}} \phi_{\mathbf{n}} + \sum_{\mathbf{n}' \prec \mathbf{n}} c_{\mathbf{n}', \mathbf{n}} \phi_{\mathbf{n}'}, \quad (18)$$

where the eigenvalue $E_{\mathbf{n}}$ is given by²⁴

$$E_{\mathbf{n}} = \sum_i (2[\mathbf{n}]_i + b + b' + 2a(N - i))^2. \quad (19)$$

From the basis (17) one can construct a set of spin wavefunctions spanning the Hilbert space of the Hamiltonian H by applying the operator Λ which projects onto states symmetric under particle permutations and reflections of the spatial coordinates, determined by the relations $K_{ij}\Lambda = P_{ij}\Lambda$, $K_i\Lambda = \Lambda$. In this way we obtain the set of spin wavefunctions

$$\psi_{\mathbf{n}, \mathbf{s}}(\mathbf{x}) = \phi(\mathbf{x}) \Lambda(e^{2i\mathbf{n} \cdot \mathbf{x}} |\mathbf{s}\rangle), \quad |\mathbf{s}\rangle \equiv |s_1, \dots, s_N\rangle. \quad (20)$$

It is clear that these wavefunctions are not linearly independent. However, using the properties of the projector Λ one can easily extract from the set of wavefunctions (20) a (non-orthonormal) basis \mathcal{B} by suitably restricting the quantum numbers \mathbf{n} and \mathbf{s} . A convenient way of achieving this end is by imposing the following conditions:

- i) $n_1 \geq n_2 \geq \dots \geq n_N \geq 0$, i.e. $\mathbf{n} \in [\mathbb{Z}^N]$.
- ii) If $n_i = n_j$, then $s_i \geq s_j$.

From Eq. (18) and the relation $H'\Lambda = H\Lambda$ one can easily check that the action of H on the basis \mathcal{B} is given by $H\psi_{\mathbf{n}, \mathbf{s}} = E_{\mathbf{n}}\psi_{\mathbf{n}, \mathbf{s}} + \text{l.o.t.}$, where l.o.t. denotes a linear combination of basis functions with quantum numbers $(\mathbf{n}', \mathbf{s}')$ satisfying $\mathbf{n}' \prec \mathbf{n}$. Thus the Hamiltonian H is again upper triangular in the basis \mathcal{B} , partially ordered according to the prescription $\psi_{\mathbf{n}, \mathbf{s}} \prec \psi_{\mathbf{n}', \mathbf{s}'}$ if $\mathbf{n} \prec \mathbf{n}'$. Its eigenvalues $E_{\mathbf{n}, \mathbf{s}} = E_{\mathbf{n}}$ are given by Eq. (19) or, taking into account that $[\mathbf{n}] = \mathbf{n}$ by condition i),

$$E_{\mathbf{n}, \mathbf{s}} = 4 \sum_i (n_i + a(\bar{\beta} + N - i))^2, \quad \bar{\beta} \equiv \frac{\beta + \beta'}{2}.$$

Writing $\mathbf{n} = (\underbrace{\nu_1, \dots, \nu_1}_{k_1}, \dots, \underbrace{\nu_r, \dots, \nu_r}_{k_r})$, it is clear that the energies $E_{\mathbf{n}, \mathbf{s}}$ have an intrinsic degeneracy $d(\mathbf{n}) = \prod_{i=1}^r \binom{m+k_i-1}{k_i}$ equal to the number of basis states $|\mathbf{s}\rangle$ compatible with condition ii) above. From the expansion

$$\frac{E_{\mathbf{n}, \mathbf{s}}}{4a} = \frac{aE_0}{4} + 2 \sum_i n_i (\bar{\beta} + N - i) + O(1/a),$$

where $a^2 E_0$ is the ground state energy of H , we obtain

$$\lim_{a \rightarrow \infty} [q^{\frac{aE_0}{4}} Z(-4aT)] = \sum_{n_1 \geq \dots \geq n_N \geq 0} d(\mathbf{n}) q^{2 \sum_i n_i (i - N - \bar{\beta})}.$$

Here

$$q \equiv e^{-1/(k_B T)},$$

where T is the temperature and k_B is Boltzmann's constant. The sum in the exponent of the RHS can be expressed in terms of ν_i and k_i as

$$\sum_{i=1}^r \nu_i k_i (2N_i + 1 - 2\bar{\beta} - 2N - k_i), \quad (21)$$

where $N_i \equiv \sum_{j=1}^i k_j$. Proceeding as in Ref. 24, we introduce the new variables $l_i = \nu_i - \nu_{i+1} > 0$ ($i = 1, \dots, r-1$) and $l_r = \nu_r \geq 0$. After a straightforward calculation one can then rewrite the sum (21) as $\sum_{j=1}^r l_j \mathcal{E}(N_j)$, where

$$\mathcal{E}(j) = j(j+1 - 2\bar{\beta} - 2N). \quad (22)$$

Denoting by \mathcal{P}_N the set of all partitions of the integer N (with order taken into account) we get the compact formula

$$\begin{aligned} \lim_{a \rightarrow \infty} [q^{\frac{aE_0}{4}} Z(-4aT)] &= \sum_{\mathbf{k} \in \mathcal{P}_N} \prod_{i=1}^r \binom{m+k_i-1}{k_i} \sum_{\substack{l_1, \dots, l_{r-1} > 0 \\ l_r \geq 0}} \prod_{j=1}^r q^{l_j \mathcal{E}(N_j)} \\ &= \frac{1}{1 - q^N} \sum_{\mathbf{k} \in \mathcal{P}_N} \prod_{i=1}^r \binom{m+k_i-1}{k_i} \cdot \prod_{j=1}^{r-1} \frac{q^{\mathcal{E}(N_j)}}{1 - q^{\mathcal{E}(N_j)}}, \end{aligned}$$

where $\mathbf{k} = (k_1, \dots, k_r)$ and we have taken into account that $N_r = N$. On the other hand, the partition function of the scalar Sutherland model H_{sc} was computed in Ref. 24, with the result

$$\lim_{a \rightarrow \infty} [q^{\frac{aE_0}{4}} Z_{\text{sc}}(-4aT)] = \prod_{i=1}^N (1 - q^{\mathcal{E}(i)})^{-1}.$$

From the last two equations and the freezing trick relation (16) one finally arrives at the following explicit formula for the partition function of the chain (4):

$$\mathcal{Z}(T) = \sum_{\mathbf{k} \in \mathcal{P}_N} \prod_{i=1}^r \binom{m+k_i-1}{k_i} \cdot q^{\sum_{i=1}^{r-1} \mathcal{E}(N_i)} \prod_{j=1}^{N-r} (1 - q^{\mathcal{E}(N'_j)}), \quad (23)$$

where $N_i = \sum_{j=1}^i k_j$,

$$\{N'_1, \dots, N'_{N-r}\} = \{1, \dots, N-1\} \setminus \{N_1, \dots, N_{r-1}\},$$

and the *dispersion relation* \mathcal{E} is given by Eq. (22).

Interestingly, the structure of the partition function (23) is the same as that of the original *closed*

Haldane–Shastry chain³⁶, albeit with a different dispersion relation depending on the single free (positive) parameter $\bar{\beta} \equiv (\beta + \beta')/2$. It was shown in Ref. 37 that a partition function of the form (23) coincides with that of a related vertex model regardless of the functional form of the dispersion relation. More precisely, consider a one-dimensional classical vertex model consisting of $N+1$ vertices connected by N intermediate bonds. Any possible state for this vertex model can be represented by a path configuration given by a vector $\mathbf{s} = (s_1, \dots, s_N)$, where $s_i \in \{1, 2, \dots, m\}$ denotes the spin state of the i -th bond. The energy function associated with this spin path configuration \mathbf{s} is defined as

$$E(\mathbf{s}) = \sum_{j=1}^{N-1} \mathcal{E}(j) \theta(s_j - s_{j+1}), \quad (24)$$

where θ is Heaviside's step function given by $\theta(x) = 0$ for $x \leq 0$ and $\theta(x) = 1$ otherwise. As shown in Ref. 37, the partition function of this vertex model is given by Eq. (23). An important consequence of this fact is that the spectrum of the $\text{su}(m)$ model (4) with sites satisfying (10), including the degeneracy of each level, is given by Eq. (24), where \mathbf{s} runs over all possible m^N spin configurations. In fact, it is well-known that the spectrum of many Yangian-invariant spin models, including the original Haldane–Shastry chain, is given by a formula of the type (24) with a suitable model-dependent dispersion relation $\mathcal{E}(j)$. This suggests that the chain (4) may also be invariant under the (untwisted) Yangian $Y(\text{gl}(m))$ for arbitrary values of β and β' .

VI. GROUND STATE AND SPECTRUM

From Eq. (24) it follows that the energy levels of the chain (4) can be computed from the formula

$$E_{\boldsymbol{\delta}} = \sum_{j=1}^{N-1} \mathcal{E}(j) \delta_j, \quad (25)$$

where each δ_j is either zero or one, and the vector $\boldsymbol{\delta} = (\delta_1, \dots, \delta_{N-1})$, which is called a *motif*, cannot contain a sequence of m or more consecutive 1's. In fact, Eq. (25) is the counterpart of Haldane's formula for the energies of the closed (antiferromagnetic) $\text{su}(m)$ Haldane–Shastry chain in terms of motifs^{13,38}, for which the dispersion relation is given by $\mathcal{E}(j) = j(j - N)$. Note, however, that Eq. (25), unlike (24), does not convey complete information on the degeneracy of each level. In this section we shall use Eq. (25) for the spectrum in terms of Haldane's motifs to compute the ground state energy and deduce an alternative expression for the distinct energy levels in terms of rapidities. This expression generalizes to arbitrary positive values of β and β' the formula conjectured in Ref. 19 for the three uniform cases.

We shall begin by computing the ground state energy. In the first place, since $\mathcal{E}(j) < 0$ for $j = 1, \dots, N-1$ and

the only restriction on the motif $\delta \equiv (\delta_1, \dots, \delta_{N-1})$ is that it cannot contain more than $m-1$ consecutive 1's, it is clear that when $m \geq N$ the ground state energy E_0 is obtained from the motif $\delta = (1^{N-1})$. Hence in this case we simply have

$$E_0 = \sum_{j=1}^{N-1} \mathcal{E}(j) = -\frac{1}{3} N(N-1)(3\bar{\beta} + 2N - 1). \quad (26)$$

On the other hand, when $m < N$ the motif that yields the minimum energy is $(1^{N_0-1}, 0, 1^{m-1}, \dots, 0, 1^{m-1})$, where $N_0 = N \bmod m$ and the 0's are in the positions $N - jm$ with $j = 1, \dots, \lfloor (N-1)/m \rfloor$. Here $\lfloor x \rfloor$ denotes the integer part of the real number x , and it is understood that if $N_0 = 0$ the first sequence $(1^{N_0-1}, 0)$ is missing. Inserting the latter motif into Eq. (25) we obtain

$$\begin{aligned} E_0 &= \sum_{j=1}^{N-1} \mathcal{E}(j) - \sum_{j=1}^{\lfloor (N-1)/m \rfloor} \mathcal{E}(N - jm) \\ &= \sum_{j=1}^{N-1} \mathcal{E}(j) - \sum_{j=1}^{(N-N_0)/m} \mathcal{E}(N - jm). \end{aligned}$$

Indeed, $\lfloor N/m \rfloor = (N - N_0)/m = \lfloor (N-1)/m \rfloor + \delta_{0,N_0}$, and the spurious term with $j = N/m$ in the last sum when $N_0 = 0$ (i.e., when N is divisible by m) is of no consequence since $\mathcal{E}(0) = 0$. Evaluating the last sum in the previous equation we easily find the following explicit formula for the ground state energy when $m < N$:

$$\begin{aligned} E_0 &= -\frac{m-1}{6m} N[4N^2 + 3(2\bar{\beta} - 1)N + m] \\ &\quad + \frac{N_0(m - N_0)}{6m} [3(2\bar{\beta} + 2N - 1) + m - 2N_0]. \end{aligned}$$

We shall next express Eq. (25) in terms of rapidities and compare the resulting formula with that conjectured in Ref. 19 for the three uniform cases. More precisely, given a motif $\delta = (\delta_1, \dots, \delta_{N-1})$ its corresponding *rapidities* are the positions r_i ($i = 1, \dots, n$) of its nonzero (i.e., 1) components, in terms of which the motif's energy is given by

$$E_\delta = \sum_{i=1}^n \mathcal{E}(r_i). \quad (27)$$

By Haldane's restriction on the $\text{su}(m)$ motifs, there cannot be more than $m-1$ consecutive rapidities. Moreover, the maximum number of rapidities in a motif, n_{\max} , is equal to the number of 1's in the ground state motif, namely $\lfloor (m-1)N/m \rfloor$. Since

$$\mathcal{E}(j) = \left(\frac{\nu}{2} - j \right)^2 - \frac{\nu^2}{4}, \quad \nu \equiv 2(\bar{\beta} + N) - 1,$$

Eq. (27) can be rewritten as

$$E_\delta = \sum_{i=1}^n \left(\rho_i^2 - \frac{\nu^2}{4} \right), \quad (28)$$

with

$$\rho_i = \frac{\nu}{2} - r_i = \bar{\beta} + N - \frac{1}{2} - r_i > 0. \quad (29)$$

Note that, unlike the rapidities r_i , the numbers ρ_i are generally not integers. From the latter equation it follows that the set of distinct energy levels of the $\text{su}(m)$ open Haldane-Shastry chain can be generated from Eq. (28) with the following two rules:

- i) The n numbers ρ_i belong to the set $\{\bar{\beta} + 1/2, \bar{\beta} + 3/2, \dots, \bar{\beta} + N - 3/2\}$, with $n = 0, 1, \dots, \lfloor (m-1)N/m \rfloor$.
- ii) There can be no more than $m-1$ consecutive ρ_i 's.

Note that the second rule is a direct consequence of Haldane's restriction on the $\text{su}(m)$ motifs, and that by definition ρ_i and ρ_j are consecutive if their difference is equal to 1. The latter description of the set of distinct energies can be easily adapted to the alternative Hamiltonian

$$\tilde{\mathcal{H}} = \sum_{i \neq j} \left(\frac{1}{|z_i - z_j|^2} + \frac{1}{|z_i + z_j|^2} \right) \mathbf{t}_i \cdot \mathbf{t}_j \quad (30)$$

considered in Ref. 19 in the three uniform cases (3). To begin with, note that from the identity (5) it follows that

$$\tilde{\mathcal{H}} = \frac{1}{2} \mathcal{H} + \frac{1}{2} \left(1 - \frac{1}{m} \right) \sum_{i \neq j} \left(\frac{1}{|z_i - z_j|^2} + \frac{1}{|z_i + z_j|^2} \right).$$

The last term coincides with the constant energy

$$\tilde{E}_0 \equiv \frac{m-1}{8m} \left[- \sum_{i \neq j} (w_{ij}^2 + w_{i\bar{j}}^2) + 2N(N-1) \right]$$

in Ref. 19, on account of the identities

$$\begin{aligned} -w_{ij}^2 &= \cot^2 \xi_{ij}^- = \sin^{-2} \xi_{ij}^- - 1 = \frac{4}{|z_i - z_j|^2} - 1, \\ -w_{i\bar{j}}^2 &= \cot^2 \xi_{ij}^+ = \sin^{-2} \xi_{ij}^+ - 1 = \frac{4}{|z_i + z_j|^2} - 1. \end{aligned}$$

Thus the distinct energies of the Hamiltonian $\tilde{\mathcal{H}}$ can be generated from the formula

$$\tilde{E} = \tilde{E}_0 + \frac{1}{2} \sum_{i=1}^n \left(\rho_i^2 - \frac{\nu^2}{4} \right) \quad (31)$$

by the two rules above for the numbers ρ_i . In particular, in the three uniform cases $\bar{\beta} = 1/2, 3/2, 1$ the ρ_i 's belong to the sets $\{1, \dots, N-1\}$, $\{2, \dots, N\}$, $\{3/2, 5/2, \dots, N-1/2\}$ and $\nu = 2N, 2(N+1), 2N+1$, respectively, so that Eq. (31) reduces to the formula conjectured in Ref. 19. Note, finally, that the constant energy \tilde{E}_0 in the previous equation can be easily computed by noting that

$$\tilde{E}_0 = \frac{m-1}{2m} \sum_{i \neq j} \left(\frac{1}{|z_i - z_j|^2} + \frac{1}{|z_i + z_j|^2} \right) = \frac{1-m}{4m} E_0,$$

where E_0 is the ground state energy of the Hamiltonian \mathcal{H} when $m \geq N$, given by Eq. (26). We thus have

$$\tilde{E}_0 = \frac{1}{12} \left(1 - \frac{1}{m} \right) N(N-1)(3\bar{\beta} + 2N - 1).$$

The fact that the spectrum of the spin chain (4) is fully described by Eqs. (22) and (24) has several important consequences that we shall now discuss. Indeed, it was shown in Refs. 39 and 40 that the level density of any quantum system whose spectrum is of the form (24) with a dispersion relation $\mathcal{E}(j)$ polynomial in j and N is normally distributed in the limit $N \rightarrow \infty$. Secondly, since an equation of the form (24) also describes the spectrum of Yangian-invariant $\text{su}(m)$ spin models, the spectrum of the chain (4) must be highly degenerate for *all* values of β and β' . In fact, from the polynomial character of this chain's dispersion relation it follows that its average degeneracy should be much higher than that of a generic Yangian-invariant model (with a non-polynomial dispersion relation). More precisely, it was shown in Ref. 41 that when $\mathcal{E}(j)$ is a polynomial in j and N the number of distinct levels is at most $O(N^{\sum_s (s+1)r(s)})$, where, for a given s , $r(s)$ is the number of monomials of the form $N^p j^s$ in \mathcal{E} . Moreover, when $\mathcal{E}(j)$ is a polynomial with *rational* coefficients the number of distinct levels is actually $O(N^{k+1})$, where k is the total degree of $\mathcal{E}(j)$ in j and N . For the dispersion relation (22) we have $r(1) = 2$, $r(2) = 1$ and $k = 2$, so that the number of distinct levels of the chain (4) is (at most) $O(N^7)$ for arbitrary $\beta + \beta'$ and $O(N^3)$ for rational $\beta + \beta'$. This is indeed much lower than for a generic Yangian-invariant spin model, for which the latter number grows exponentially⁴¹ with N .

VII. CONCLUSIONS AND OUTLOOK

In this paper we have introduced an integrable generalization of the $\text{su}(m)$ Simons–Altshuler open chain^{19,21,22} depending on two arbitrary positive parameters β and β' , whose sites are determined by the zeros of a suitable Jacobi polynomial. Using the results in Ref. 19, we have shown that this model's ground state can be obtained from the chiral correlator of the $c = m - 1$ free boson boundary CFT. We have computed the first few nontrivial conserved charges stemming from the model's twisted Yangian symmetry, and evaluated the chains' partition function in closed form for arbitrary values of its parameters. From the partition function we have been able to deduce a formula for the energy spectrum in terms of Haldane's motifs, with a dispersion relation similar to that of the original (closed) Haldane–Shastry chain. As shown in Ref. 42, this formula can be applied to derive the chain's thermodynamical properties, which could be relevant in the context of the single-impurity Kondo problem. Finally, it should be noted that the chain's connection to a conformal field theory could be exploited in

several different ways. For instance, in the $\text{su}(2)$ case it is well known that the spin correlation functions of this type of models satisfy a system of linear equations whose coefficients depend on the chain sites in a simple way¹⁹. This fact, which was used in the latter reference to compute the correlators in the first uniform case, provides a promising way for evaluating the correlators in the $\text{su}(2)$ case for arbitrary values of the parameters β and β' .

ACKNOWLEDGMENTS

The authors would like to thank H.-H. Tu for his helpful comments on a previous version of this manuscript. This work was partially supported by Spain's MINECO under grant no. FIS2015-63966, and by the Universidad Complutense de Madrid and Banco Santander under grant no. GR3/14-910556.

Appendix: Summation formula for the zeros of Jacobi polynomials

In this appendix we shall prove the identity

$$\begin{aligned} \sum_{j(\neq i)} (w_{ij} - w_{i\bar{j}})^2 &= \frac{1}{3} (\beta - \beta') (2 - \beta - \beta') w_{i0}^2 \\ &+ \frac{4}{3} \beta' (2 - \beta') w_{i1}^2 - \frac{4N^2}{3} + \frac{4N}{3} (4 - \beta - \beta') \\ &- \frac{2}{3} (1 + \beta') (\beta - \beta') + \frac{8\beta}{3} - 4, \end{aligned} \quad (\text{A.1})$$

which is used in Section IV to simplify the conserved quantities of the open Haldane–Shastry chain. To begin with, note that

$$u_j = \text{Re } z_j = \frac{1}{2} (z_j + z_j^{-1}) = \frac{z_j^2 + 1}{2z_j},$$

and hence

$$\begin{aligned} u_i - u_j &= \frac{(z_j - z_i)(1 - z_i z_j)}{2z_i z_j}, \\ w_{ij} - w_{i\bar{j}} &= \frac{z_i + z_j}{z_i - z_j} - \frac{z_i + z_j^{-1}}{z_i - z_j^{-1}} = \frac{z_j^2 - 1}{z_j} (u_i - u_j)^{-1}. \end{aligned}$$

Taking into account that $z_j = e^{2i\xi_j}$, so that

$$\left(\frac{1 - z_j^2}{z_j} \right)^2 = \left(\frac{1}{z_j} - z_j \right)^2 = -4 \sin^2(2\xi_j) = -4(1 - u_j^2),$$

we have

$$\sum_{j(\neq i)} (w_{ij} - w_{i\bar{j}})^2 = -4 \sum_{j(\neq i)} \frac{1 - u_j^2}{(u_i - u_j)^2}. \quad (\text{A.2})$$

Since

$$\frac{1 - u_j^2}{(u_i - u_j)^2} = \frac{1 - u_i^2}{(u_i - u_j)^2} + \frac{2u_i}{u_i - u_j} - 1,$$

the sum in the RHS of Eq. (A.2) can be evaluated using Eqs. (5.2a)-(5.2b) in Ref. 26, which in our notation read

$$\begin{aligned} 2(1 - u_i^2) \sum_{j(\neq i)} (u_i - u_j)^{-1} &= \beta - \beta' + (\beta + \beta')u_i, \\ 12(1 - u_i^2)^2 \sum_{j(\neq i)} (u_i - u_j)^{-2} &= 4(N - 1)(N + \beta + \beta') \\ &\quad - (\beta - \beta')^2 - 2(\beta - \beta')(4 + \beta + \beta')u_i \\ &\quad - [4N(N + \beta + \beta' - 1) + (\beta + \beta')(4 + \beta + \beta')]u_i^2. \end{aligned}$$

Using the latter formulas and the identities

$$\begin{aligned} w_{ii}^2 &= \left(\frac{z_i^2 + 1}{z_i^2 - 1} \right)^2 = -\cot^2(2\xi_i) = -\frac{u_i^2}{1 - u_i^2}, \\ w_{i0}^2 &= \left(\frac{z_i + 1}{z_i - 1} \right)^2 = -\cot^2 \xi_i = \frac{u_i + 1}{u_i - 1}, \end{aligned}$$

from which it follows that

$$\frac{u_i}{1 - u_i^2} = \frac{1}{1 - u_i} - \frac{1}{1 - u_i^2} = \frac{1}{2} (2w_{ii}^2 - w_{i0}^2 - 1),$$

we obtain Eq. (A.1) after a straightforward calculation.

* bireswar.basumallick@saha.ac.in

† ffinkel@ucm.es

‡ Corresponding author. Email address: artemio@ucm.es

- ¹ P. Schauß, J. Zeiher, T. Fukuhara, S. Hild, M. Chenau, T. Macrì, T. Pohl, I. Bloch, and C. Gross, *Science* **347**, 1455 (2015).
- ² P. Richerme, Z.-X. Gong, A. Lee, C. Senko, J. Smith, M. Foss-Feig, S. Michalakakis, A. V. Gorshkov, and C. Monroe, *Nature* **511**, 198 (2014).
- ³ P. Jurcevic, B. P. Lanyon, P. Hauke, C. Hempel, P. Zoller, R. Blatt, and C. F. Roos, *Nature* **511**, 202 (2014).
- ⁴ K. Kim, M.-S. Chang, R. Islam, S. Korenblit, L.-M. Duan, and C. Monroe, *Phys. Rev. Lett.* **103**, 120502 (2009).
- ⁵ D. Porras and J. I. Cirac, *Phys. Rev. Lett.* **92**, 207901 (2004).
- ⁶ W. Lechner and P. Zoller, *Phys. Rev. Lett.* **111**, 185306 (2013).
- ⁷ H. Weimer, R. Löw, T. Pfau, and H. P. Büchler, *Phys. Rev. Lett.* **101**, 250601 (2008).
- ⁸ F. D. M. Haldane, *Phys. Rev. Lett.* **60**, 635 (1988).
- ⁹ B. S. Shastri, *Phys. Rev. Lett.* **60**, 639 (1988).
- ¹⁰ F. Gebhard and D. Vollhardt, *Phys. Rev. Lett.* **59**, 1472 (1987).
- ¹¹ C. Gros, R. Joynt, and T. M. Rice, *Phys. Rev. B* **36**, 381 (1987).
- ¹² F. D. M. Haldane, *Phys. Rev. Lett.* **67**, 937 (1991).
- ¹³ F. D. M. Haldane, Z. N. C. Ha, J. C. Talstra, D. Bernard, and V. Pasquier, *Phys. Rev. Lett.* **69**, 2021 (1992).
- ¹⁴ D. Bernard, M. Gaudin, F. D. M. Haldane, and V. Pasquier, *J. Phys. A: Math. Gen.* **26**, 5219 (1993).
- ¹⁵ J. I. Cirac and G. Sierra, *Phys. Rev. B* **81**, 104431 (2010).
- ¹⁶ A. E. B. Nielsen, J. I. Cirac, and G. Sierra, *J. Stat. Mech.-Theory E* **2011**, P11014 (2011).
- ¹⁷ A. E. B. Nielsen, J. I. Cirac, and G. Sierra, *Phys. Rev. Lett.* **108**, 257206 (2012).
- ¹⁸ R. Bondesan and T. Quella, *Nucl. Phys. B* **886**, 483 (2014).
- ¹⁹ H.-H. Tu and G. Sierra, *Phys. Rev. B* **92**, 041119(R) (2015).
- ²⁰ Here and in what follows sums and products over the indices i, j, k, l range from 1 to N unless otherwise stated, and $\sum_{i \neq j}$, $\sum_{i(\neq j)}$ respectively denote summation over the two indices i, j and over the single index i with the restriction $i \neq j$.

- ²¹ B. D. Simons and B. L. Altshuler, *Phys. Rev. B* **50**, 1102 (1994).
- ²² D. Bernard, V. Pasquier, and D. Serban, *Europhys. Lett.* **30**, 301 (1995).
- ²³ T. Yamamoto, *Phys. Lett. A* **208**, 293 (1995).
- ²⁴ A. Enciso, F. Finkel, A. González-López, and M. A. Rodríguez, *Nucl. Phys. B* **707**, 553 (2005).
- ²⁵ E. Corrigan and R. Sasaki, *J. Phys. A: Math. Gen.* **35**, 7017 (2002).
- ²⁶ S. Ahmed, M. Bruschi, F. Calogero, M. A. Olshanetsky, and A. M. Perelomov, *Nuovo Cimento B* **49**, 173 (1979).
- ²⁷ A. B. J. Kuijlaars and W. Van Assche, *J. Approx. Theory* **99**, 167 (1999).
- ²⁸ G. Szegő, *Orthogonal Polynomials*, 4th ed. (Amer. Math. Soc., Providence, RI, 1975).
- ²⁹ E. K. Sklyanin, *J. Phys. A: Math. Gen.* **21**, 2375 (1988).
- ³⁰ Unless otherwise specified, here and in what follows sums over the $\mathfrak{su}(m)$ indices a, b, c range from 1 to $m^2 - 1$.
- ³¹ A. J. Macfarlane, A. Sudbery, and P. H. Weisz, *Commun. Math. Phys.* **11**, 77 (1968).
- ³² Actually, Eq. (8) of Ref. 19 contains a minor typo, i.e., the term $w_{ik} + w_{i\bar{k}}$ in the last sum should be $w_{jk} + w_{j\bar{k}}$.
- ³³ A. P. Polychronakos, *Phys. Rev. Lett.* **70**, 2329 (1993).
- ³⁴ B. Sutherland and B. S. Shastri, *Phys. Rev. Lett.* **71**, 5 (1993).
- ³⁵ A. P. Polychronakos, *Nucl. Phys. B* **419**, 553 (1994).
- ³⁶ F. Finkel and A. González-López, *Phys. Rev. B* **72**, 174411 (2005).
- ³⁷ B. Basu-Mallick, N. Bondyopadhyaya, and K. Hikami, *SIGMA* **6**, 091 (2010).
- ³⁸ F. D. M. Haldane, in *Correlation Effects in Low-dimensional Electron Systems*, Springer Series in Solid-state Sciences, Vol. 118, edited by A. Okiji and N. Kawakami (1994) pp. 3–20.
- ³⁹ A. Enciso, F. Finkel, and A. González-López, *Phys. Rev. E* **82**, 051117 (2010).
- ⁴⁰ P. Banerjee and B. Basu-Mallick, *J. Math. Phys.* **53**, 083301 (2012).
- ⁴¹ F. Finkel and A. González-López, *Ann. Phys.-New York* **361**, 520 (2015).

⁴² A. Enciso, F. Finkel, and A. González-López, Ann. Phys.-New York **327**, 2627 (2012).